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Sharp constant for a 2D anisotropic Sobolev inequality with critical nonlinearity

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ABSTRACT

For the 2-dimensional anisotropic Sobolev inequality of the form

$$\int_{\mathbb{R}^2} |u|^6 \, dx\,dy \leqslant \alpha \left(\int_{\mathbb{R}^2} u_x^2 \, dx\,dy \right)^2 \int_{\mathbb{R}^2} \left|D_x^{-1} u_y\right|^2 dx\,dy,$$

it is proved that the sharp (smallest) positive constant α is exactly as $3(\int_{\mathbb{R}^2} \phi_x^2 \, dx\,dy)^{-2}$, where ϕ is a minimal action solution of $(u_{xx} + |u|^4 u)_x = D_x^{-1} u_{yy}$.

1. Introduction

In this paper, we use variational methods to study the sharp (smallest) positive constant α in the 2-dimensional anisotropic Sobolev inequality [2, p. 323] with critical nonlinearity

$$\int_{\mathbb{R}^2} |u|^6 \, dx\,dy \leqslant \alpha \left(\int_{\mathbb{R}^2} u_x^2 \, dx\,dy \right)^2 \int_{\mathbb{R}^2} \left|D_x^{-1} u_y\right|^2 dx\,dy, \quad u \in Y_0, \tag{1.1}$$

where Y_0 is the closure of $\partial_x(C_0^\infty(\mathbb{R}^2)) := \{g_x\colon\ g \in C_0^\infty(\mathbb{R}^2)\}$ under the norm

$$\|u\|_{Y_0}^2 = \int_{\mathbb{R}^2} \left(u_x^2 + \left|D_x^{-1} u_y\right|^2\right) dx\,dy,$$

where we denote $u_x = \frac{\partial u(x,y)}{\partial x}$, $u_y = \frac{\partial u(x,y)}{\partial y}$ and define D_x^{-1} by

$$D_x^{-1} h(x,y) = \int_{-\infty}^{x} h(s,y)\, ds.$$

Then Y_0 is a Hilbert space with an induced inner product defined by $\langle u, v\rangle = \int_{\mathbb{R}^2} (u_x v_x + D_x^{-1} u_y D_x^{-1} v_y)\, dx\,dy$. We say (1.1) is an inequality with critical nonlinearity, because (1.1) is the limit case of the following well-known anisotropic Sobolev inequality [2, p. 323]: for $0 < p < 4$, there is a positive constant $C > 0$ such that

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$$\int_{\mathbb{R}^2} |u|^{p+2}\,dx\,dy \leqslant C\left(\int_{\mathbb{R}^2} u_x^2\,dx\,dy\right)^{\frac{p}{2}}\left(\int_{\mathbb{R}^2} \left|D_x^{-1}u_y\right|^2\,dx\,dy\right)^{\frac{p}{4}}\left(\int_{\mathbb{R}^2} u^2\,dx\,dy\right)^{\frac{4-p}{4}} \tag{1.2}$$

holds for all $u \in Y_1$, where Y_1 is the closure of $\{g_x\colon\ g \in C_0^\infty(\mathbb{R}^2)\}$ under the norm

$$\|u\|_{Y_1}^2 = \int_{\mathbb{R}^2} \left(u_x^2 + \left|D_x^{-1}u_y\right|^2 + u^2\right)dx\,dy.$$

It is known that inequality (1.2) has been used extensively in the study of the generalized Kadomtsev–Petviashvili equation. For example, de Bouard et al. [4] proved that for $0 < p < 4$, there is a nontrivial solitary wave solution of

$$\left(u_t + u_{xxx} + \left(u^{p+1}\right)_x\right)_x = u_{yy}, \quad (x, y) \in \mathbb{R}^2,\ t > 0. \tag{1.3}$$

While for $p \geqslant 4$, de Bouard et al. [4] proved that (1.3) did not possess nontrivial solitary wave solution. Some decaying properties of solitary wave of (1.3) were obtained in [5]. In [9,10], Liu et al. studied the stability and instability of solitary waves of (1.3) by using (1.2). The sharp (smallest) value of C in (1.2) and its applications have been obtained in [6,7]. We also refer the interested reader to Weinstein [11], where the best constant for the Gagliardo–Nirenberg inequality was proved.

Inequality (1.1) is the limit case of the inequality (1.2) and we believe that the study of the sharp constant α in (1.1) is not without interest. The main results of the present paper are the following Theorems 1.1 and 1.2.

Theorem 1.1. *The sharp* (*smallest*) *positive constant* α *in* (1.1) *is exactly as*

$$\alpha = 3\left(\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy\right)^{-2}, \tag{1.4}$$

where ϕ *is a minimal action solution of*

$$\left(u_{xx} + |u|^4 u\right)_x = D_x^{-1}u_{yy}, \quad u \in Y_0 \text{ and } u \neq 0. \tag{1.5}$$

Here and after, by saying ϕ is a minimal action solution of (1.5) we mean that ϕ is a solution of (1.5) and a minimizer of

$$d = \inf\left\{S(u)\colon\ u \in \Gamma\right\}, \tag{1.6}$$

where

$$S(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2}u_x^2 + \frac{1}{2}\left|D_x^{-1}\partial_y u\right|^2 - \frac{1}{6}|u|^6\right)dx\,dy, \qquad \Gamma = \left\{u \in Y_0\colon\ u \neq 0 \text{ and } I(u) = 0\right\} \quad \text{with}$$

$$I(u) = \int_{\mathbb{R}^2} \left(u_x^2 + \left|D_x^{-1}\partial_y u\right|^2 - |u|^6\right)dx\,dy.$$

Remark. The uniqueness of the minimal action solution ϕ of (1.5) is still an *open* problem. But the following theorem implies that α is independent of the choice of the minimal action solution ϕ.

Theorem 1.2. *Let* d *be defined as in* (1.6)*. Then* $\alpha = \frac{3}{4d^2}$.

In order to prove Theorems 1.1 and 1.2, we use variational methods. Our strategy is as follows. In the first place, we solve the minimization problem (1.6) and prove that there is $\phi \in \Gamma$ such that $d = S(\phi)$ and ϕ is a minimal action solution of (1.5). In the second place, we determine the minimum of the following minimizing problem

$$C_0 = \inf\left\{J(u)\colon\ u \neq 0 \text{ and } u \in Y_0\right\}, \tag{1.7}$$

where

$$J(u) = \left(\int_{\mathbb{R}^2} u_x^2\,dx\,dy\right)^2 \int_{\mathbb{R}^2} \left|D_x^{-1}u_y\right|^2\,dx\,dy \left(\int_{\mathbb{R}^2} |u|^6\,dx\,dy\right)^{-1}.$$

Then $\alpha = C_0^{-1}$ is the sharp constant such that (1.1) holds. In the third place, we use the properties of the minimal action solution ϕ to prove Theorem 1.2.

Remark. In [11], Weinstein studied the best constant C_G in the Gagliardo–Nirenberg inequality

$$\int_{\mathbb{R}^N} |u|^{q+1} \leqslant C_G\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{\frac{N(q-1)}{4}}\left(\int_{\mathbb{R}^N} |u|^2\right)^{\frac{2(q+1)-N(q-1)}{4}}, \quad u \in W^{1,2}(\mathbb{R}^N).$$

This C_G was determined directly from solving the following minimization problem

$$C_G^{-1} = \inf\left\{\frac{(\int_{\mathbb{R}^N} |\nabla u|^2)^{\frac{N(q-1)}{4}}(\int_{\mathbb{R}^N} |u|^2)^{\frac{2(q+1)-N(q-1)}{4}}}{\int_{\mathbb{R}^N} |u|^{q+1}},\ u \neq 0,\ u \in W^{1,2}(\mathbb{R}^N)\right\},$$

due to the compactness embedding of $W^{1,2}_{radial}(\mathbb{R}^N)$ into $L^{q+1}(\mathbb{R}^N)$ for $1 < q < 2^* - 1$, where

$$W^{1,2}_{radial}(\mathbb{R}^N) = \{u \in W^{1,2}(\mathbb{R}^N):\ u(x) = u(|x|)\},$$

$2^* = 2N/(N-2)$ for $N \geqslant 3$ and $2^* = +\infty$ for $N = 2$. However, for the minimization problem considered here, we are facing an anisotropic Sobolev space Y_0 and we cannot use the above compact embedding to study the minimization problem (1.7). We believe that the method presented here may have independent interests and can be used to study the best constant of other kind of inequalities.

The rest of the paper is organized as follows. In Section 2, we study the minimization problem (1.6) and give some properties of the minimal action solution ϕ of (1.5). Section 3 is devoted to the study of the minimum C_0 in (1.7) and the proofs of Theorem 1.1 and Theorem 1.2.

2. Minimal action solution of (1.5)

In this section, we study the minimal action solutions of (1.5) and its properties. Firstly, we need the following lemmas.

Lemma 2.1. *For any $u \in Y_0$ and $u \neq 0$, there is a unique $\theta_u > 0$ such that $\theta_u u \in \Gamma$. Moreover, if $I(u) < 0$ then $0 < \theta_u < 1$.*

Proof. For $u \neq 0$ and any $\theta > 0$, we have that

$$S(\theta u) = \int_{\mathbb{R}^2}\left(\frac{\theta^2}{2}u_x^2 + \frac{\theta^2}{2}\left|D_x^{-1}\partial_y u\right|^2 - \frac{\theta^6}{6}|u|^6\right)dx\,dy.$$

Direct computations arrive at

$$\theta_u = \left(\int_{\mathbb{R}^2}\left(u_x^2 + \left|D_x^{-1}\partial_y u\right|^2\right)dx\,dy\right)^{\frac{1}{4}}\left(\int_{\mathbb{R}^2}|u|^6\,dx\,dy\right)^{-\frac{1}{4}}.$$

From the expression of $I(u)$, one deduces that if $I(u) < 0$, i.e.,

$$\int_{\mathbb{R}^2}\left(u_x^2 + \left|D_x^{-1}\partial_y u\right|^2\right)dx\,dy < \int_{\mathbb{R}^2}|u|^6\,dx\,dy,$$

then $0 < \theta_u < 1$. $\square$

Lemma 2.2. *$\Gamma \neq \emptyset$ and Γ is a manifold. Moreover there exists $\rho > 0$ such that for any $u \in \Gamma$, $\|u\|_{Y_0} \geqslant \rho > 0$.*

Proof. $\Gamma \neq \emptyset$ follows from the previous lemma. For any $u \in \Gamma$,

$$\langle I'(u), u\rangle = 2\int_{\mathbb{R}^2}\left(u_x^2 + \left|D_x^{-1}\partial_y u\right|^2\right)dx\,dy - 6\int_{\mathbb{R}^2}|u|^6\,dx\,dy = -4\int_{\mathbb{R}^2}|u|^6\,dx\,dy < 0,$$

which implies that Γ is a manifold. Next, for any $u \in \Gamma$ using inequality (1.1) and Young inequality, we know that there is a positive constant C such that

$$\int_{\mathbb{R}^2}\left(u_x^2 + \left|D_x^{-1}\partial_y u\right|^2\right)dx\,dy = \int_{\mathbb{R}^2}|u|^6\,dx\,dy \leqslant C\left(\int_{\mathbb{R}^2}\left(u_x^2 + \left|D_x^{-1}\partial_y u\right|^2\right)dx\,dy\right)^3.$$

Hence $\|u\|_{Y_0} \geqslant C^{-\frac{1}{4}} := \rho > 0$. The proof is complete. $\square$

Lemma 2.3. *If $v \in \Gamma$ and $S(v) = d$, then v is a critical point of S on Y_0, i.e. $S'(v) = 0$.*

Proof. By Lagrangian multiplier rule, we know there is $\mu \in \mathbb{R}$ such that

$$S'(v) = \mu I'(v).$$

According to $\langle S'(v), v\rangle = I(v) = 0$ and

$$\left\langle I'(v), v\right\rangle = -4 \int_{\mathbb{R}^2} |u|^6\, dx\, dy < 0,$$

we have that $\mu = 0$. Therefore $S'(v) = 0$. The proof is complete. □

Theorem 2.4. *We have $d > 0$ and there is a $\phi \in \Gamma$ such that $d = S(\phi)$. Moreover ϕ is a minimal action solution of* (1.5)*.*

Proof. The fact $d > 0$ follows from Lemma 2.2. We start with proving that there is $\phi \in \Gamma$ such that $d = S(\phi)$. Let $\{u_n\}_{n\in\mathbb{N}} \subset \Gamma$ be a minimizing sequence of the minimization problem (1.6), i.e.

$$u_n \neq 0, \qquad I(u_n) = 0 \quad \text{and} \quad d + o(1) = S(u_n).$$

We obtain from Lemma 2.2 that there is a positive constant C such that $\|u_n\|_{Y_0} \leqslant C$ and

$$\liminf_{n\to\infty} \int_{\mathbb{R}^2} |u_n|^6\, dx\, dy > 0.$$

Note that for any $(x, y) \in \mathbb{R}^2$,

$$S\big(u(\cdot + x, \cdot + y)\big) = S(u) \quad \text{and} \quad I\big(u(\cdot + x, \cdot + y)\big) = I(u);$$

and for any $\lambda > 0$,

$$S\big(\lambda u\big(\lambda^2 x, \lambda^4 y\big)\big) = S(u) \quad \text{and} \quad I\big(\lambda u\big(\lambda^2 x, \lambda^4 y\big)\big) = I(u),$$

we obtain from concentration compactness lemma of Lions [8] (see also [1]) that there are $\lambda_n > 0$ and $(x_n, y_n) \in \mathbb{R}^2$ such that

$$\varphi_n(x, y) := \lambda_n u_n\big(\lambda_n^2(x + x_n), \lambda_n^4(y + y_n)\big)$$

satisfies $\|\varphi_n\|_{Y_0} = \|u_n\|_{Y_0} \leqslant C$, $\int_{\mathbb{R}^2} |\varphi_n|^6\, dx\, dy = \int_{\mathbb{R}^2} |u_n|^6\, dx\, dy$,

$$I(\varphi_n) = I(u_n) \quad \text{and} \quad S(\varphi_n) = S(u_n).$$

Moreover $\varphi_n \rightharpoonup \phi \neq 0$ weakly in Y_0 and $\varphi_n \to \phi$ a.e. in $\mathbb{R}^2$.

If $I(\phi) < 0$, then by Lemma 2.1 there is a $0 < \theta_\phi < 1$ such that $\theta_\phi \phi \in \Gamma$. Therefore using Fatou lemma and $I(\varphi_n) = 0$, we obtain that

$$\begin{aligned} d + o(1) = S(\varphi_n) &= \left(\frac{1}{2} - \frac{1}{6}\right) \int_{\mathbb{R}^2} |\varphi_n|^6\, dx\, dy \geqslant \frac{1}{3} \int_{\mathbb{R}^2} |\phi|^6\, dx\, dy + o(1) \\ &= \frac{1}{3}\theta_\phi^{-6} \int_{\mathbb{R}^2} |\theta_\phi \phi|^6\, dx\, dy + o(1) = \theta_\phi^{-6} S(\theta_\phi \phi) + o(1). \end{aligned}$$

It is deduced from $0 < \theta_\phi < 1$ that $d > S(\theta_\phi \phi)$ which is a contradiction because $\theta_\phi \phi \in \Gamma$.

If $I(\phi) > 0$, then using Brezis–Lieb lemma [3] one has

$$0 = I(\varphi_n) = I(\phi) + I(v_n) + o(1),$$

where $v_n = \varphi_n - \phi$. $I(\phi) > 0$ implies that

$$\limsup_{n\to\infty} I(v_n) < 0. \tag{2.1}$$

From Lemma 2.1 we know that there are $t_n := \theta_{v_n}$ such that $t_n v_n \in \Gamma$. Moreover we claim that $\limsup_{n\to\infty} t_n \in (0, 1)$. Indeed if $\limsup_{n\to\infty} t_n = 1$, then there is a subsequence $\{t_{n_k}\}$ such that $\lim_{k\to\infty} t_{n_k} = 1$. Therefore from $t_{n_k} v_{n_k} \in \Gamma$ one has that

$$I(v_{n_k}) = I(t_{n_k} v_{n_k}) + o(1) = o(1).$$

This contradicts (2.1). Hence $\limsup_{n\to\infty} t_n \in (0,1)$. Since

$$d+o(1)=\frac{1}{3}\int\limits_{\mathbb{R}^2}|\varphi_{n_k}|^6\,dx\,dy \geqslant \frac{1}{3}\int\limits_{\mathbb{R}^2}|v_{n_k}|^6\,dx\,dy+o(1) \geqslant \frac{1}{3}t_{n_k}^{-6}\int\limits_{\mathbb{R}^2}|t_{n_k}v_n|^6\,dx\,dy+o(1),$$

one has $d > S(t_{n_k}v_{n_k})$, which is a contradiction.

Thus $I(\phi)=0$. Using Brezis–Lieb lemma again, one gets $\|v_n\|_{Y_0}\to 0$, i.e. $\varphi_n \to \phi$ in Y_0. Indeed, if $\|v_n\|_{Y_0}\not\to 0$ then we have two cases: (i) For $\int |v_n|^6\,dx\,dy \not\to 0$, we have

$$d+o(1)=S(\varphi_n)=S(\phi)+S(v_n)+o(1)\geqslant d+d+o(1),$$

which it is a contradiction; (ii) For $\int |v_n|^6\,dx\,dy \to 0$, we have

$$d+o(1)=S(\varphi_n)=S(\phi)+\frac{1}{2}\|v_n\|_{Y_0}^2+o(1)>d,$$

which it is also a contradiction. Therefore, $\|v_n\|_{Y_0}\to 0$ and we conclude that $S(\varphi_n)\to S(\phi)$ and $d=S(\phi)$.

Next, by Lemma 2.3, we know ϕ is a minimal action solution of (1.5). The proof is complete. □

Now we give some properties of the minimal action solution ϕ.

Lemma 2.5. *Let ϕ be a minimal action solution of* (1.5)*. Then*

$$\int\limits_{\mathbb{R}^2}\left(\phi_x^2-\frac{2}{3}|\phi|^6\right)dx\,dy=0. \tag{2.2}$$

Proof. The proof is similar to [6, Lemma 2.2]. We omit the details here. □

Lemma 2.6. *Let ϕ be a minimal action solution of* (1.5)*. Then*

$$\int\limits_{\mathbb{R}^2}|\phi|^6\,dx\,dy=\frac{3}{2}\int\limits_{\mathbb{R}^2}\phi_x^2\,dx\,dy, \qquad \int\limits_{\mathbb{R}^2}\left|D_x^{-1}\partial_y\phi\right|^2dx\,dy=\frac{1}{2}\int\limits_{\mathbb{R}^2}\phi_x^2\,dx\,dy.$$

Proof. Since ϕ is a minimal action solution of (1.5), one has $I(\phi)=0$. Combining this with (2.2), one easily get the conclusion. □

We conclude this section with another characterization of the minimal action solution ϕ of (1.5). Define

$$T(u)=\int\limits_{\mathbb{R}^2}\left(u_x^2+\left|D_x^{-1}\partial_y u\right|^2\right)dx\,dy$$

and, for $r>0$, set

$$T_r=\inf\left\{T(u):\ u\in Y_0 \text{ and } \int\limits_{\mathbb{R}^2}|u|^6\,dx\,dy=r\right\}.$$

Then we have:

Proposition 2.7. *Let ϕ be a minimal action solution of* (1.5)*. Then ϕ is a minimizer of T_r with $r=\int_{\mathbb{R}^2}|\phi|^6\,dx\,dy$.*

Proof. Since ϕ is a minimal action solution of (1.5), one has

$$S(\phi)\leqslant S(u)$$

for any $u\in Y_0$, $u\neq 0$ and $I(u)=0$. Denote

$$T_{r_0}=\inf\left\{T(u):\ u\in Y_0 \text{ and } \int\limits_{\mathbb{R}^2}|u|^6\,dx\,dy=\int\limits_{\mathbb{R}^2}|\phi|^6\,dx\,dy\right\}.$$

One has $T(\phi)\geqslant T_{r_0}$. Next, we will prove that for any $u\in Y_0$ satisfying $\int_{\mathbb{R}^2}|u|^6\,dx\,dy=\int_{\mathbb{R}^2}|\phi|^6\,dx\,dy$, there holds

$$T(\phi)\leqslant T(u).$$

In the first place, for any $\mu > 0$,

$$I(\mu u) = \mu^2 T(u) - \mu^6 \int_{\mathbb{R}^2} |u|^6 \, dx\, dy.$$

Hence

$$\mu_0 = \big(T(u)\big)^{\frac{1}{4}} \bigg(\int_{\mathbb{R}^2} |u|^6 \, dx\, dy \bigg)^{-\frac{1}{4}}$$

is such that $I(\mu_0 u) = 0$. In the second place, $\mu_0 u \neq 0$ implies that

$$\begin{aligned} S(\phi) \leqslant S(\mu_0 u) &= \frac{1}{2}\mu_0^2 T(u) - \frac{1}{6}\mu_0^6 \int_{\mathbb{R}^2} |u|^6 \, dx\, dy \\ &= \frac{1}{3}\big(T(u)\big)^{\frac{3}{2}} \bigg(\int_{\mathbb{R}^2} |u|^6 \, dx\, dy \bigg)^{-\frac{1}{2}} = \frac{1}{3}\big(T(u)\big)^{\frac{3}{2}} \bigg(\int_{\mathbb{R}^2} |\phi|^6 \, dx\, dy \bigg)^{-\frac{1}{2}}. \end{aligned}$$

Since $\int_{\mathbb{R}^2} |\phi|^6 \, dx\, dy = T(\phi)$ and $S(\phi) = \frac{1}{3} T(\phi)$, one has

$$\big(T(\phi)\big)^{\frac{3}{2}} \leqslant \big(T(u)\big)^{\frac{3}{2}},$$

i.e., $T(\phi) \leqslant T(u)$. Since u is arbitrary, one obtains $T(\phi) \leqslant T_{r_0}$.

Therefore $T(\phi) = T_{r_0}$ and hence ϕ is a minimizer of T_r with $r = \int_{\mathbb{R}^2} |\phi|^6 \, dx\, dy$. The proof is complete. □

3. Sharp constant α

After studying the minimal action solution ϕ of (1.5), we are now in a position to determine the exact value of the sharp constant α. The key step is to determine the exact value C_0.

Proof of Theorem 1.1. The proof is divided into two steps. In the first step, we prove that

$$C_0 \geqslant \frac{1}{3} \bigg(\int_{\mathbb{R}^2} \phi_x^2 \, dx\, dy \bigg)^2. \tag{3.1}$$

For any $u \in Y_0$ and $u \neq 0$, we define $w(x, y) = \lambda u(\mu x, \xi y)$. Then one gets from direct computations that

$$\begin{aligned} &\int_{\mathbb{R}^2} w_x^2 \, dx\, dy = \lambda^2 \mu \xi^{-1} \int_{\mathbb{R}^2} u_x^2 \, dx\, dy, \\ &\int_{\mathbb{R}^2} \big|D_x^{-1} \partial_y w\big|^2 \, dx\, dy = \lambda^2 \mu^{-3} \xi \int_{\mathbb{R}^2} \big|D_x^{-1} \partial_y u\big|^2 \, dx\, dy \quad \text{and} \\ &\int_{\mathbb{R}^2} |w|^6 \, dx\, dy = \lambda^6 \mu^{-1} \xi^{-1} \int_{\mathbb{R}^2} |u|^6 \, dx\, dy. \end{aligned}$$

Define

$$\lambda^2 \mu^{-3} \xi \int_{\mathbb{R}^2} \big|D_x^{-1} \partial_y u\big|^2 \, dx\, dy = \frac{1}{2} \int_{\mathbb{R}^2} \phi_x^2 \, dx\, dy \tag{3.2}$$

and

$$\lambda^6 \mu^{-1} \xi^{-1} \int_{\mathbb{R}^2} |u|^6 \, dx\, dy = \frac{3}{2} \int_{\mathbb{R}^2} \phi_x^2 \, dx\, dy. \tag{3.3}$$

Then (3.2) and (3.3) implies that

$$\lambda^4 \mu^2 \xi^{-2} \int_{\mathbb{R}^2} |u|^6 \, dx\, dy = 3 \int_{\mathbb{R}^2} \big|D_x^{-1} \partial_y u\big|^2 \, dx\, dy. \tag{3.4}$$

Therefore

$$\int_{\mathbb{R}^2} w_x^2\,dx\,dy = \left(3\int_{\mathbb{R}^2} \left|D_x^{-1}\partial_y u\right|^2 dx\,dy \left(\int_{\mathbb{R}^2} |u|^6\,dx\,dy\right)^{-1}\right)^{\frac{1}{2}} \int_{\mathbb{R}^2} u_x^2\,dx\,dy. \tag{3.5}$$

The definition of w and (3.2)–(3.3) imply that

$$T(w) \geqslant T(\phi),$$

and from the expression of $T(w)$, it is deduced that

$$\int_{\mathbb{R}^2} w_x^2\,dx\,dy \geqslant \int_{\mathbb{R}^2} \phi_x^2\,dx\,dy.$$

Combining this with (3.5) and the definition of J we get that

$$J(u) \geqslant \frac{1}{3}\left(\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy\right)^2.$$

Since $u \neq 0$ is arbitrary, one has $C_0 \geqslant \frac{1}{3}(\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy)^2$.

In the second step, we prove that $C_0 \leqslant \frac{1}{3}(\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy)^2$. Indeed, from $\phi \neq 0$ and

$$\begin{aligned} J(\phi) &= \left(\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy\right)^2 \int_{\mathbb{R}^2} \left|D_x^{-1}\partial_y \phi\right|^2 dx\,dy \left(\int_{\mathbb{R}^2} |\phi|^6\,dx\,dy\right)^{-1} \\ &= \left(\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy\right)^2 \left(\frac{1}{2}\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy\right)\left(\frac{3}{2}\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy\right)^{-1} \\ &= \frac{1}{3}\left(\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy\right)^2, \end{aligned}$$

we immediately have $C_0 \leqslant \frac{1}{3}(\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy)^2$. Thus

$$\alpha = C_0^{-1} = 3\left(\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy\right)^{-2}.$$

The proof is complete. □

Remark. Observing the proof, we know that to get (3.1), it is essential to solve λ, μ and ξ from (3.2) and (3.3) (*usually one cannot solve three variables only from two equations*).

Proof of Theorem 1.2. Since $d = S(\phi)$ and ϕ is a minimal action solution of (1.5), we obtain from Lemma 2.6 that

$$d = \frac{1}{2}\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^2} \left|D_x^{-1}\partial_y \phi\right|^2 dx\,dy - \frac{1}{6}\int_{\mathbb{R}^2} |\phi|^6\,dx\,dy = \left(\frac{1}{2} + \frac{1}{4} - \frac{1}{4}\right)\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy = \frac{1}{2}\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy.$$

Therefore $\alpha = C_0^{-1} = 3(\int_{\mathbb{R}^2} \phi_x^2\,dx\,dy)^{-2} = 3/(4d^2)$. □

Remark. There are anisotropic Sobolev inequalities like (1.1) in higher dimensions $\mathbb{R}^N$ with $N \geqslant 3$, e.g. see [2], for which the sharp constant of the related inequality is still unknown. However, since here we use the special structure of the problem in 2-dimensional case, it should be further investigated if the proposed method can be extended to such kind of inequalities on higher dimensions.

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